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# On the regularity over positively graded algebras

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**Abstract**

We study the relationship between the Tor-regularity and the local-regularity over a positively graded algebra defined over a field which coincide if the algebra is a standard graded polynomial ring. In this case both are characterizations of the so-called Castelnuovo–Mumford regularity. Moreover, we can characterize a standard graded polynomial ring as a $K$-algebra with extremal properties with respect to the Tor- and the local-regularity. For modules of finite projective dimension we get a nice formula relating the two regularity notions. Interesting examples are given to help to understand the relationship between the Tor- and the local-regularity in general.



*Keywords:* Regularity; Positively graded algebras; Koszul algebras; Linear resolutions; Local cohomology; Free resolutions

## 1. Introduction

Let $K$ be a field and $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring with unique graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Eisenbud and Goto [7] proved that for a finitely generated graded $S$-module $M$ the finite numbers

(1) $\inf\{r \in \mathbb{Z}\colon$ for all $i \geqslant 0$ and all $s > r$ we have $\operatorname{Tor}_i^S(M, K)_{i+s} = 0\}$,
(2) $\inf\{r \in \mathbb{Z}\colon$ for all $i \geqslant 0$ and all $s > r$ we have $H^i_{\mathfrak{m}}(M)_{s-i} = 0\}$

coincide. Usually one calls this number the *Castelnuovo–Mumford*-regularity $\operatorname{reg}_S(M)$ of $M$. Moreover, if we denote by $M_{\geqslant q}$ for $q \geqslant 0$ the truncation of $M$ defined as the graded $S$-module with homogeneous components: $(M_{\geqslant q})_i = M_i$ if $i \geqslant q$ and $(M_{\geqslant q})_i = 0$ for $i < q$, then $\operatorname{reg}_S(M)$

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is also the least number $q$ such that $M_{\geqslant q}$ is non-trivial and has a $q$-linear $S$-resolution, i.e. $\dim_K \operatorname{Tor}_i^S(M_{\geqslant q}, K)_{i+j} = 0$ for $j \neq q$.

It is a natural question to understand the relationship between these numbers in the situation where the $K$-algebra is not longer a polynomial ring. In the following a *positively graded $K$-algebra* $R$ is a Noetherian commutative $K$-algebra such that $R = \bigoplus_{i \geqslant 0} R_i$ with $R_0 = K$. We denote always by $\mathfrak{m} = \bigoplus_{i>0} R_i$ the unique graded maximal ideal of $R$. We say that $R$ is *standard graded* if $R$ is generated in degree 1. If $R$ is a polynomial ring, then we call $R$ a positively graded polynomial ring and standard graded polynomial ring respectively. A finitely generated graded $R$-module is always a non-trivial $\mathbb{Z}$-graded $R$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$. The crucial definitions of this paper are:

**Definition 1.1.** Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module. Then:

(i) $\operatorname{reg}_R^T(M) = \inf\{r \in \mathbb{Z}:$ for all $i \geqslant 0$ and all $s > r$ we have $\operatorname{Tor}_i^R(M, K)_{i+s} = 0\}$ is called the *Tor-regularity* of $M$.
(ii) $\operatorname{reg}_R^L(M) = \inf\{r \in \mathbb{Z}:$ for all $i \geqslant 0$ and all $s > r$ we have $H_{\mathfrak{m}}^i(M)_{s-i} = 0\}$ is called the *local-regularity* of $M$.

After some preliminary remarks in Section 2 we prove in Section 3 our first main result:

**Theorem 1.2.** *Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module. Then*:

(i) $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) \leqslant \operatorname{reg}_R^T(M)$.
(ii) *If $R$ is standard graded, then* $\operatorname{reg}_R^T(M) \leqslant \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$.

Observe that the upper inequality is essentially due to Avramov and Eisenbud [1]. Jørgensen [11] proved a much more generally version for complexes over not necessarily commutative $K$-algebras which have a balanced dualizing complex. For modules we present here a straight forward proof which avoids the technical machinery used in [11]. See Herzog and Restuccia [10] for a similar result over standard graded $K$-algebras. Note that by a result of Avramov and Eisenbud [1] if $R$ is a Koszul algebra, i.e. $\operatorname{reg}_R^T(K) = 0$ where we consider $K = R/\mathfrak{m}$ naturally as an $R$-module, it is still true that $\operatorname{reg}_R^T(M)$ is the least number $q$ such that $M_{\geqslant q}$ is non-trivial and has a $q$-linear $R$-resolution.

Having certain inequalities of invariants related to a module $M$, it is of course interesting to understand for which modules equality holds. Considering standard graded $K$-algebras $R$ we know by the graded version of the famous result of Auslander–Buchsbaum–Serre that $R$ is a polynomial ring if and only if $\operatorname{pd}_R(M) < \infty$ for all finitely generated graded $R$-modules. Moreover, it is enough to show that $\operatorname{pd}_R(K) < \infty$ to conclude that $R$ is a polynomial ring. Interestingly a polynomial ring is also characterized by extremal properties with respect to the regularity notions introduced above. More precisely, in Section 4 we prove:

**Theorem 1.3.** *Let $R$ be a standard graded $K$-algebra. The following statements are equivalent*:

(i) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$;
(ii) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$;

(iii) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M)$;
(iv) *$R$ is Koszul and* $\operatorname{reg}_R^L(R) = 0$;
(v) *$R = K[x_1, \dots, x_n]$ is a standard graded polynomial ring.*

In the general case we can still show the following nice fact:

**Theorem 1.4.** *Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module such that* $\operatorname{pd}_R(M) < \infty$. *Then*

$$\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M).$$

Also see Chardin [5] for similar results. By giving an example that the converse of the latter result does not hold, it still interesting to understand for which modules we have $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$ and $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$ respectively. We conclude the paper in Section 5 with the observation that there exists a Koszul algebra $R$ such that $\operatorname{depth}(R) > 0$ and $r = \operatorname{reg}_R^L(R) > 0$ and we have for $0 < j < r$ that

$$0 = \operatorname{reg}_R^L(\mathfrak{m}^j) - r < \operatorname{reg}_R^T(\mathfrak{m}^j) = j < r = \operatorname{reg}_R^L(\mathfrak{m}^j).$$

In this sense any number between $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R)$ and $\operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$ can be the Tor-regularity of a module.

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## 2. Preliminaries

In this section we fix some further notation and recall some definitions. For facts related to commutative algebra we refer to the book of Eisenbud [6]. A standard reference on homological algebra is Weibel [14]. Now following Priddy [12] we define:

**Definition 2.1.** Let $R$ be a standard graded $K$-algebra. Then $R$ is called a *Koszul algebra* if $\operatorname{reg}_R^T(K) = 0$ where we consider $K = R/\mathfrak{m}$ naturally as an $R$-module.

For a positively graded $K$-algebra $R$ and a finitely generated graded $R$-module $M$ we say that $M$ has a *$q$-linear resolution* if $\operatorname{Tor}_i^R(M, K)_{i+j} = 0$ for $j \neq q$. Thus if we consider the minimal graded free resolution

$$F_\bullet : \cdots \to F_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} F_0 \to M \to 0$$

of $M$ with $F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{ij}^R(M)}$ where $\beta_{ij}^R(M) = \dim_K \operatorname{Tor}_l^R(M, K)_j$ are the *graded Betti-numbers* of $M$, then $M$ has a $q$-linear resolution if and only if $\beta_{ii+j}^R(M) = 0$ for $j \neq q$. In particular, if $R$ is standard graded, then $R$ is Koszul if and only if $K$ has a 0-linear resolution. E.g. a standard graded polynomial ring $K[x_1, \dots, x_n]$ is trivially a Koszul algebra since the Koszul complex on the variables $x_1, \dots, x_n$ provides a minimal graded free resolution of $K$ which is 0-linear. We will need the following result which is essentially due to Avramov and Eisenbud [1]:

**Theorem 2.2.** *Let $R$ be a Koszul algebra and $M$ be a finitely generated graded $R$-module, then*

$$\operatorname{reg}^T_R(M) \leqslant \operatorname{reg}^L_R(M) < \infty.$$

**Proof.** Let $R = S/I$ where $S$ is a standard graded polynomial ring and $I \subset S$ is a graded ideal containing no linear forms. Avramov and Eisenbud proved that $\operatorname{reg}^T_R(M) \leqslant \operatorname{reg}^T_S(M)$. But over a polynomial ring we have $\operatorname{reg}^T_S(M) = \operatorname{reg}^L_S(M)$ by Eisenbud and Goto [7]. Moreover, it is well known that $\operatorname{reg}^L_S(M) = \operatorname{reg}^L_R(M)$ simply because the local cohomology of $M$ with respect to the maximal ideal computed over $S$ is isomorphic to the local cohomology of $M$ with respect to the maximal ideal computed over $R$. That $\operatorname{reg}^L_R(M) < \infty$ follows now from the fact that $\operatorname{reg}^L_S(M) < \infty$. (E.g. see [4]: There are only finitely many local cohomology groups not zero and all of them have the property that $H^i_{\mathfrak{m}}(M)_j = 0$ for $j \gg 0$.) □

The Koszul property cannot be decided by knowing only finitely many graded Betti numbers of $K$. (See [13] for examples.) Recently Avramov and Peeva [2] proved the following remarkable result:

**Theorem 2.3.** *Let $R$ be a positively graded $K$-algebra. Then the following statements are equivalent*:

(i) *$R$ is Koszul*;
(ii) *$R$ is standard graded and for every finitely generated graded $R$-module $M$ we have* $\operatorname{reg}^T_R(M) < \infty$;
(iii) *$R$ is standard graded and we have* $\operatorname{reg}^T_R(K) < \infty$.

Thus $K$ is a test-module for the Koszul property using the invariant $\operatorname{reg}^T_R(K)$. In the next sections we will use occasionally the following observations:

**Remark 2.4.** Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module.

(i) We have $\operatorname{reg}^L_R(M) < \infty$. Indeed, the arguments used in the proof of Theorem 2.2 to show that $\operatorname{reg}^L_R(M)$ is finite, did not used the fact that $R$ is Koszul.
(ii) $\operatorname{reg}^L_R(K) = 0$ because $H^i_{\mathfrak{m}}(K) = 0$ for $i \neq 0$ and $H^0_{\mathfrak{m}}(K) = K$.
(iii) By a result of Grothendieck (e.g. see [4]) we know that

$$H^i_{\mathfrak{m}}(M) \begin{cases} = 0 & \text{for } i < \operatorname{depth}(M) \text{ and } i > \dim(M), \\ \neq 0 & \text{for } i = \operatorname{depth}(M) \text{ and } i = \dim(M). \end{cases}$$

## 3. Comparison of the Tor- and the local-regularity

We want to compare the notion of regularities as introduced in Section 2. The main result of this section is the next theorem.

**Theorem 3.1.** *Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module. Then*:

(i) $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) \leqslant \operatorname{reg}_R^T(M)$;
(ii) *if $R$ is standard graded, then* $\operatorname{reg}_R^T(M) \leqslant \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$.

**Proof.** (i): If $\operatorname{reg}_R^T(M) = \infty$, then nothing is to show. Next assume that $\operatorname{reg}_R^T(M) < \infty$. Observe that the numbers $\operatorname{reg}_R^L(M)$ and $\operatorname{reg}_R^L(R)$ are always finite by Remark 2.4. We consider the minimal graded free resolution

$$F_\bullet : \cdots \to F_l \xrightarrow{\partial_l} \cdots \xrightarrow{\partial_1} F_0 \to M \to 0$$

of $M$ with $F_l = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{lj}^R(M)}$ where $\beta_{lj}^R(M) = \dim_K \operatorname{Tor}_l^R(M, K)_j$ are the graded Betti-numbers of $M$. Note that $\beta_{lj}^R(M) = 0$ for $j > l + \operatorname{reg}_R^T(M)$. Define

$$C_l := \operatorname{Ker} \partial_l \quad \text{for } l \geqslant 0 \text{ and set } C_{-1} := M.$$

For $l \geqslant 0$ the short exact sequences

$$0 \to C_l \to F_l \to C_{l-1} \to 0$$

give rise to a long exact local cohomology sequence in degree $j - i$

$$\begin{aligned} 0 &\to H_\mathfrak{m}^0(C_l)_{j-i} \to H_\mathfrak{m}^0(F_l)_{j-i} \to H_\mathfrak{m}^0(C_{l-1})_{j-i} \to \cdots \\ &\to H_\mathfrak{m}^i(C_l)_{j-i} \to H_\mathfrak{m}^i(F_l)_{j-i} \to H_\mathfrak{m}^i(C_{l-1})_{j-i} \to \cdots. \end{aligned}$$

Since $H_\mathfrak{m}^i(R(-j)) = H_\mathfrak{m}^i(R)(-j)$ and $H_\mathfrak{m}^i(\cdot)$ is an additive functor, we have that for $j > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M) + l$ and for all $i \geqslant 0$ that

$$H_\mathfrak{m}^i(F_l)_{j-i} = 0.$$

Thus for $l = 0$ and $j > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M)$ we obtain

$$H_\mathfrak{m}^i(M)_{j-i} \cong H_\mathfrak{m}^{i+1}(C_0)_{j-i}.$$

For $j + 1 > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M) + 1 \Leftrightarrow j > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M)$ we get analogously

$$H_\mathfrak{m}^{i+1}(C_0)_{j-i} \cong H_\mathfrak{m}^{i+2}(C_1)_{j-i}.$$

Using an appropriate induction we see that for $j > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M)$ we have

$$H_\mathfrak{m}^i(M)_{j-i} \cong \cdots \cong H_\mathfrak{m}^{i+l+1}(C_l)_{j-i}.$$

Note that $\dim C_l \leqslant \dim R =: d$ and we get from Remark 2.4 that for $l \geqslant d - i$ we have $H_\mathfrak{m}^{i+l+1}(C_l) = 0$. All in all we obtain for $j > \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M)$ and $l \geqslant d - i$ that

$$H_\mathfrak{m}^i(M)_{j-i} \cong H_\mathfrak{m}^{i+l+1}(C_l)_{j-i} = 0.$$

Hence $\operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M) \geqslant \operatorname{reg}_R^L(M)$ as desired.

(ii): If $\operatorname{reg}_R^T(K) = \infty$, then nothing is to show. Next assume $\operatorname{reg}_R^T(K) < \infty$. Then it follows from Theorem 2.3 that $\operatorname{reg}_R^T(K) = 0$ and $R$ is a Koszul algebra. But now the inequality $\operatorname{reg}_R^T(M) \leqslant \operatorname{reg}_R^L(M)$ was shown in Theorem 2.2. □

In Section 4 we will see that most times $\operatorname{reg}_R^L(M) \neq \operatorname{reg}_R^T(M)$, so these two regularities do no coincide in general. For Koszul algebras we still have the result that the regularity is related to linear resolutions of truncations of $M$. (See [1,7,11].) Here for a graded $R$-module $M$ and an integer $q$ we define the truncation $M_{\geqslant q}$ of $M$ as the graded $R$-module with homogeneous components:

$$(M_{\geqslant q})_i = \begin{cases} M_i & \text{if } i \geqslant q, \\ 0 & \text{else.} \end{cases}$$

**Theorem 3.2.** *Let $R$ be a Koszul algebra, $M$ be a finitely generated graded $R$-module and $q \in \mathbb{Z}$. The following statements are equivalent*:

(i) $q \geqslant \operatorname{reg}_R^T(M)$;
(ii) $\operatorname{Tor}_i^R(M, K)_{i+j} = 0$ *for all $i \geqslant 0$ and all $j > q$*;
(iii) $M_{\geqslant q}$ *has a $q$-linear $R$-resolution.*

*In particular, $\operatorname{reg}_R^T(M)$ is the least $q \in \mathbb{Z}$ such that $M_{\geqslant q}$ is non-trivial and has a $q$-linear free resolution. Moreover, if $q \geqslant \operatorname{reg}_R^T(M)$, then $M_{\geqslant q}$ has a $q$-linear free resolution.*

**Proof.** The equivalence of (i) and (ii) follows directly from the definition of $\operatorname{reg}_R^T(M)$.

Let now $F_\bullet$ be a minimal graded free resolution of $K$ as an $R$-module. Since $R$ is Koszul we have $\operatorname{reg}_R^T(K) = 0$ and thus $0 = \dim_K \operatorname{Tor}_i^R(K, K)_{i+j} = \beta_{i,i+j}^R(K)$ for $j \neq 0$. Hence

$$F_\bullet : \cdots \to R(-i)^{c_i} \to \cdots \to R^{c_0} \to K \to 0.$$

Assume (ii) holds. The $K$-vector space $\operatorname{Tor}_i^R(M, K)_{i+j}$ is the $i$th homology of the following complex:

$$F_\bullet \otimes_R M : \cdots \to \big(R(-i)^{c_i} \otimes_R M\big)_{i+j} \to \cdots \to \big(R^{c_0} \otimes_R M\big)_{i+j} \to 0.$$

For $j > q$ we have $(R(-i)^{c_i} \otimes_R M)_{i+j} = (R(-i)^{c_i} \otimes_R M_{\geqslant q})_{i+j}$. It follows that for $j > q$ we get

$$0 = \operatorname{Tor}_i^R(M, K)_{i+j} = H_i(M \otimes_R F_\bullet)_{i+j} = H_i(M_{\geqslant q} \otimes_R F_\bullet)_{i+j} = \operatorname{Tor}_i^R(M_{\geqslant q}, K)_{i+j}.$$

Since for $j < q$ we have $(M_{\geqslant q})_j = 0$, we get that $(M_{\geqslant q} \otimes_R R(-i)^{c_i})_{i+j} = 0$, and thus $\operatorname{Tor}_i^R(M_{\geqslant q}, K)_{i+j} = 0$ for $j < q$. All in all we proved (iii).

Assume (iii) holds. The computation above shows that for integers $j > q$ we have that $\operatorname{Tor}_i^R(M, K)_{i+j} = \operatorname{Tor}_i^R(M_{\geqslant q}, K)_{i+j} = 0$, which shows (ii). This concludes the proof. □

## 4. The borderline cases

It is a natural question to characterize the situations where we have equalities $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$ and $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$ respectively. Over a standard graded $K$-algebra the cases that these equalities hold for all finitely generated graded $R$-modules are easily described. In fact, Eisenbud and Goto [7] proved that $\operatorname{reg}_R^L(M) = \operatorname{reg}_R^T(M)$ for all finitely generated graded $R$-modules $M$ if $R$ is a standard graded polynomial ring. The next theorem shows that a standard graded polynomial ring is the only standard graded $K$-algebra with this property. This results extends also in the module case an observation in [11, Corollary 2.8].

**Theorem 4.1.** *Let $R$ be a standard graded $K$-algebra. The following statements are equivalent*:

(i) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$;
(ii) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$;
(iii) *for all finitely generated graded $R$-modules $M$ we have* $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M)$;
(iv) *$R$ is Koszul and* $\operatorname{reg}_R^L(R) = 0$;
(v) *$R = K[x_1, \dots, x_n]$ is a standard graded polynomial ring.*

**Proof.** (iv) $\Rightarrow$ (i), (ii), (iii): Assume that $R$ is Koszul and $\operatorname{reg}_R^L(R) = 0$. Since $R$ is Koszul we have that $\operatorname{reg}_R^T(K) = 0$ by Theorem 2.3. Let $M$ be a finitely generated graded $R$-module. It follows from Theorem 3.1 that

$$\operatorname{reg}_R^L(M) = \operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) \leqslant \operatorname{reg}_R^T(M) \leqslant \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K) = \operatorname{reg}_R^L(M).$$

Hence $\operatorname{reg}_R^T(M) = \operatorname{reg}_L^T(M)$ in this case. Thus (i), (ii) and (iii) hold.

(i) $\Rightarrow$ (iv): Assume that for all finitely generated graded $R$-modules $M$ we have that $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$. For $M = K$ we get that $\operatorname{reg}_R^T(K) = \operatorname{reg}_R^L(K) - \operatorname{reg}_R^L(R) < \infty$. It follows from Theorem 2.3 that $\operatorname{reg}_R^T(K) = 0$. Thus $R$ is Koszul and $\operatorname{reg}_R^L(R) = \operatorname{reg}_R^L(K) - \operatorname{reg}_R^T(K) = 0$ where the last equality follows from Remark 2.4.

(ii) $\Rightarrow$ (iv): Now we assume that for all finitely generated graded $R$-modules $M$ we have $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K)$. For $M = R$ we get that $0 = \operatorname{reg}_R^T(R) = \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(K)$. In particular, $\operatorname{reg}_R^T(K) < \infty$. It follows again from Theorem 2.3 that $\operatorname{reg}_R^T(K) = 0$. Hence $R$ is Koszul and $\operatorname{reg}_R^L(R) = -\operatorname{reg}_R^T(K) = 0$.

(iii) $\Rightarrow$ (iv): This is shown analogously to the proof of "(ii) $\Rightarrow$ (iv)."

(v) $\Rightarrow$ (iv): If $R = K[x_1, \dots, x_n]$ is a standard graded polynomial ring, then $R$ is of course Koszul because the Koszul complex provides a linear free resolution for the $R$-module $K$. But we also know $H_{\mathfrak{m}}^i(R) = 0$ for $i \neq n$ and $H_{\mathfrak{m}}^n(R) \cong K[x_1^{-1}, \dots, x_n^{-1}](n)$ as $\mathbb{Z}$-graded $R$-modules. Hence $\operatorname{reg}_R^L(R) = 0$.

(iv) $\Rightarrow$ (v): Next we assume that $R$ is Koszul and $\operatorname{reg}_R^L(R) = 0$. Let $R = S/I$ where $S = K[x_1, \dots, x_n]$ is a standard graded polynomial ring and $I \subset S$ is a graded ideal. We also denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of $S$ and without loss of generality we assume that $I \subseteq \mathfrak{m}^2$ contains no linear forms. Since the local cohomology of $R$ with respect to $\mathfrak{m}$ as an $R$-module is isomorphic to the local cohomology of $R$ with respect to $\mathfrak{m}$ as an $S$-module, we have $\operatorname{reg}_S^L(R) = 0$. For finitely generated graded $S$-modules $M$ we know already that $\operatorname{reg}_S^L(M) = \operatorname{reg}_S^T(M)$ by what we have proved so far. (Use (v) $\Rightarrow$ (iv) and the equivalence of (iii) and (iv).)

Hence $\operatorname{reg}_S^T(R) = 0$. But then it follows that $I = (0)$ is the only possibility and thus $R = S$ is a standard graded polynomial ring. This concludes the proof. $\square$

The latter result shows that for a standard graded $K$-algebra the borderline cases of Theorem 3.1 hold for all finitely generated graded modules only over a polynomial ring. But it is still a natural question to characterize for an arbitrary positively graded $K$-algebra which modules have extremal properties with respect to the bounds in Theorem 3.1. Surprisingly we have that for graded modules of finite projective dimension always the lower inequality of Theorem 3.1 is an equality.

**Theorem 4.2.** *Let $R$ be a positively graded $K$-algebra and $M$ be a finitely generated graded $R$-module such that* $\operatorname{pd}_R(M) < \infty$. *Then*

$$\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M).$$

**Proof.** We prove the assertion by induction on $\operatorname{pd}_R(M)$. Assume first that $\operatorname{pd}_R(M) = 0$, then there exist finitely many $a_i \in \mathbb{Z}$ such that

$$0 \to \bigoplus_i R(-a_i) \to M \to 0$$

is a minimal graded free resolution of $M$ over $R$. It follows from the definition of $\operatorname{reg}_R^T$ that

$$\operatorname{reg}_R^T(M) = \max\{a_i\}.$$

Moreover, we see that

$$H_{\mathfrak{m}}^i(M)_{k-i} = H_{\mathfrak{m}}^i\Big(\bigoplus_j R(-a_j)\Big)_{k-i} = \bigoplus_j H_{\mathfrak{m}}^i(R)_{k-a_j-i},$$

and thus as desired

$$\operatorname{reg}_R^L(M) = \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M).$$

Assume now $0 < \operatorname{pd}_R(M) < \infty$. Let $F_0$ be the first graded free module in the minimal graded free resolution of $M$ over $R$ and let $G_1$ be the kernel of the map $F_0 \to M$. Thus we have the short exact sequence

$$(*) \quad 0 \to G_1 \to F_0 \to M \to 0.$$

We have $\operatorname{pd}_R(G_1) = \operatorname{pd}_R(M) - 1$ and hence we can apply the induction hypotheses to $G_1$:

$$\operatorname{reg}_R^L(G_1) = \operatorname{reg}_R^T(G_1) + \operatorname{reg}_R^L(R).$$

Because of the definitions of the minimal graded free resolution of $M$ and of $\operatorname{reg}_R^T$ we have

$$\operatorname{reg}_R^T(G_1) \leqslant \operatorname{reg}_R^T(M) + 1.$$

Let $F_0 = \bigoplus_i R(-a_i)$ and

$$a := \max\{a_i\}.$$

Thus $\operatorname{reg}_R^L(F_0) = a + \operatorname{reg}_R^L(R)$ and $a \leqslant \operatorname{reg}_R^T(M)$. Now we have to distinguish three cases:

(a) $\operatorname{reg}_R^T(G_1) = \operatorname{reg}_R^T(M) + 1$: For $\operatorname{reg}_R^L(G_1)$ there exists an integer $j \in \mathbb{Z}$ such that $H_{\mathfrak{m}}^j(G_1)_{\operatorname{reg}_R^L(G_1)-j} \neq 0$. It follows from $(*)$ that

$$\cdots \to H_{\mathfrak{m}}^{j-1}(M)_{\operatorname{reg}_R^L(G_1)-1-(j-1)} \to H_{\mathfrak{m}}^j(G_1)_{\operatorname{reg}_R^L(G_1)-j} \to H_{\mathfrak{m}}^j(F_0)_{\operatorname{reg}_R^L(G_1)-j} \to \cdots.$$

Since

$$\begin{aligned}\operatorname{reg}_R^L(G_1) &= \operatorname{reg}_R^T(G_1) + \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M) + 1 + \operatorname{reg}_R^L(R)\\ &\geqslant a + 1 + \operatorname{reg}_R^L(R) > a + \operatorname{reg}_R^L(R)\end{aligned}$$

we have $H_{\mathfrak{m}}^j(F_0)_{\operatorname{reg}_R^L(G_1)-j} = 0$. Now

$$H_{\mathfrak{m}}^{j-1}(M)_{\operatorname{reg}_R^L(G_1)-1-(j-1)} \neq 0$$

because $H_{\mathfrak{m}}^{j-1}(M)_{\operatorname{reg}_R^L(G_1)-1-(j-1)}$ maps surjective to $H_{\mathfrak{m}}^j(G_1)_{\operatorname{reg}_R^L(G_1)-j} \neq 0$. We get

$$\operatorname{reg}_R^L(M) \geqslant \operatorname{reg}_R^L(G_1) - 1 = \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(G_1) - 1 = \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M).$$

By Theorem 3.1 we know already

$$\operatorname{reg}_R^L(M) \leqslant \operatorname{reg}_R^L(R) + \operatorname{reg}_R^T(M).$$

Thus we have equality and the desired assertion follows in this case.

(b) $\operatorname{reg}_R^T(G_1) < \operatorname{reg}_R^T(M)$: For the number $a$ as defined as above we have

$$a = \operatorname{reg}_R^T(M) > \operatorname{reg}_R^T(G_1) = \operatorname{reg}_R^L(G_1) - \operatorname{reg}_R^L(R).$$

For the number $\operatorname{reg}_R^L(F_0)$ there exists an $j \in \mathbb{Z}$ such that $H_{\mathfrak{m}}^j(F_0)_{\operatorname{reg}_R^L(F_0)-j} \neq 0$. By $(*)$ we have the exact sequence

$$\cdots \to H_{\mathfrak{m}}^j(G_1)_{\operatorname{reg}_R^L(F_0)-j} \to H_{\mathfrak{m}}^j(F_0)_{\operatorname{reg}_R^L(F_0)-j} \to H_{\mathfrak{m}}^j(M)_{\operatorname{reg}_R^L(F_0)-j} \to \cdots.$$

Now

$$\operatorname{reg}_R^L(F_0) = a + \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M) + \operatorname{reg}_R^L(R) > \operatorname{reg}_R^T(G_1) + \operatorname{reg}_R^L(R) = \operatorname{reg}_R^L(G_1).$$

Thus $H_{\mathfrak{m}}^j(G_1)_{\operatorname{reg}_R^L(F_0)-j} = 0$ and

$$H_{\mathfrak{m}}^j(M)_{\operatorname{reg}_R^L(F_0)-j} \neq 0$$

since $H^j_{\mathfrak{m}}(F_0)_{\operatorname{reg}^L_R(F_0)-j}$ maps injective into $H^j_{\mathfrak{m}}(M)_{\operatorname{reg}^L_R(F_0)-j}$. We obtain

$$\operatorname{reg}^L_R(M) \geqslant \operatorname{reg}^L_R(F_0) = a + \operatorname{reg}^L_R(R) = \operatorname{reg}^T_R(M) + \operatorname{reg}^L_R(R).$$

It follows again from Theorem 3.1 that

$$\operatorname{reg}^L_R(M) \leqslant \operatorname{reg}^L_R(R) + \operatorname{reg}^T_R(M).$$

Hence we have equality and the assertion follows in case (b).

(c) $\operatorname{reg}^T_R(G_1) = \operatorname{reg}^T_R(M)$: We have for $a$ as defined as above that

$$a = \operatorname{reg}^T_R(M) = \operatorname{reg}^T_R(G_1).$$

For the number $\operatorname{reg}^L_R(F_0)$ there exists an integer $j \in \mathbb{Z}$ such that $H^j_{\mathfrak{m}}(F_0)_{\operatorname{reg}^L_R(F_0)-j} \neq 0$. More precisely, if we write $F_0 = R(-a) \oplus F'_0$ for some graded free $R$-module $F'_0$, then we can assume that

$$H^j_{\mathfrak{m}}\big(R(-a)\big)_{\operatorname{reg}^L_R(F_0)-j} \neq 0$$

and the induced projection map

$$\tau_1 : H^j_{\mathfrak{m}}(F_0)_{\operatorname{reg}^L_R(F_0)-j} \to H^j_{\mathfrak{m}}\big(R(-a)\big)_{\operatorname{reg}^L_R(F_0)-j}$$

is surjective. By $(*)$ we have the exact sequence

$$\cdots \to H^j_{\mathfrak{m}}(G_1)_{\operatorname{reg}^L_R(F_0)-j} \to H^j_{\mathfrak{m}}(F_0)_{\operatorname{reg}^L_R(F_0)-j} \to H^j_{\mathfrak{m}}(M)_{\operatorname{reg}^L_R(F_0)-j} \to \cdots.$$

If

$$\tau_2 : H^j_{\mathfrak{m}}(G_1)_{\operatorname{reg}^L_R(F_0)-j} \to H^j_{\mathfrak{m}}(F_0)_{\operatorname{reg}^L_R(F_0)-j}$$

would not be surjective, then $H^j_{\mathfrak{m}}(M)_{\operatorname{reg}^L_R(F_0)-j} \neq 0$ and it follows that

$$\operatorname{reg}^L_R(M) \geqslant \operatorname{reg}^L_R(F_0) = a + \operatorname{reg}^L_R(R) = \operatorname{reg}^T_R(M) + \operatorname{reg}^L_R(R).$$

Again we know from Theorem 3.1 that

$$\operatorname{reg}^L_R(M) \leqslant \operatorname{reg}^L_R(R) + \operatorname{reg}^T_R(M)$$

and thus we have the desired equality.

It remains to show that indeed $\tau_2$ is not surjective. Assume for a moment that $\tau_2$ is surjective. Then also the composed map

$$\tau_3 = \tau_1 \circ \tau_2 : H^j_{\mathfrak{m}}(G_1)_{\operatorname{reg}^L_R(F_0)-j} \to H^j_{\mathfrak{m}}\big(R(-a)\big)_{\operatorname{reg}^L_R(F_0)-j}$$

would be surjective. In particular, $\tau_3$ is not the zero map. Now we consider again the short exact sequence

$$0 \to G_1 \xrightarrow{\varphi_1} F_0 \to M \to 0.$$

Since $F_0$ was the first module in the minimal graded free resolution of $M$, we have that $\operatorname{Im}\varphi_1 \subseteq \mathfrak{m}F_0$. Since $\operatorname{reg}_R^T(G_1) = \operatorname{reg}_R^T(M) = a$, we know that $G_1$ is generated in degrees $\leqslant a$. But then for any generator $x$ of $G_1$, it is not possible that $\varphi_1(x)$ involves the free generator corresponding to $R(-a)$ in $F_0$. In other words, if we compose $\varphi_1$ with the natural projection map $\varphi_2 : F_0 \to R(-a)$, then the induced map $\varphi_3 : G_1 \to R(-a)$ is the zero map.

Next we observe that the maps $\tau_1$, $\tau_2$ and $\tau_3$ are induced by $\varphi_2, \varphi_1$ and $\varphi_3$. Indeed, consider the modified Cech-complex $C_\bullet$ (e.g. see [4, p. 130]). Then for some graded $R$-modules $W, W'$ and a homogeneous map $\psi : W \to W'$ we have that $H_\mathfrak{m}^j(W) = H^i(W \otimes_R C_\bullet)$, and the natural map $H_\mathfrak{m}^j(W) \to H_\mathfrak{m}^j(W')$ corresponds to $\psi \otimes_R C_\bullet$. This implies that the map $\tau_3$ has to be the zero map, because already $\varphi_3$ is the zero map. Thus we have a contradiction. This concludes the proof. □

Now one could hope the converse of Theorem 4.2 is also true. But this is not the case as the next example shows.

**Example 4.3.** Let $K[x, y]$ be a standard graded polynomial ring in 2 variables and consider

$$R = K[x, y]/\big(x^2, xy, y^2\big).$$

Then $R$ is a Koszul algebra since its defining ideal is a monomial ideal generated in degree 2 (see [9]). $R$ is zero-dimensional and thus Cohen–Macaulay. Let $\omega_R$ be the graded canonical module of $R$. Then we have

$$\operatorname{reg}_R^L(\omega_R) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(\omega_R),$$

but $\operatorname{pd}_R(\omega_R) = \infty$.

**Proof.** In the following we identify ideals of $R$ and $S$. Let $\mathfrak{m} = (x, y)$ be the maximal ideal of $R$. We have $\mathfrak{m} = \sqrt{(0)}$ and $\mathfrak{m}^2 = 0$. For a graded $K$-vector space $W$ we set

$$s(W) = \max\{i \in \mathbb{Z}\colon\ W_i \neq 0\}.$$

Since $R$ is zero-dimensional and thus also $\omega_R$ is zero-dimensional, it follows from Remark 2.4 that $R = H_\mathfrak{m}^0(R)$, $\omega_R = H_\mathfrak{m}^0(\omega_R)$, and for $i > 0$ that $H_\mathfrak{m}^i(R) = H_\mathfrak{m}^i(\omega_R) = 0$. Hence

$$\operatorname{reg}_R^L(R) = s(R) \quad \text{and} \quad \operatorname{reg}_R^L(\omega_R) = s(\omega_R).$$

By the definition of $R$ we have $s(R) = 1$. By graded local duality we know

$$\omega_R = \operatorname{Hom}_K(R, K)$$

with $(\omega_R)_i = \operatorname{Hom}_K(R_{-i}, K)$. Thus we see that $\dim_K(\omega_R)_{-1} = 2$, $\dim_K(\omega_R)_0 = 1$ and $\dim_K(\omega_R)_i = 0$ for $i \neq -1, 0$. Hence $s(\omega_R) = 0$ and we have

$$\operatorname{reg}_R^L(\omega_R) - \operatorname{reg}_R^L(R) = -1,$$

$\omega_R$ is a faithful module, thus not all generators of $\omega_R$ can be annihilated. It follows that $\omega_R$ is generated in degree $-1$ with 2 minimal generators. The minimal graded free resolution of $\omega_R$ starts with

$$\cdots \to R^2(+1) \to \omega_R \to 0.$$

Since $\mathfrak{m}^2 = 0$ in $R$ and the matrices corresponding to the maps in a minimal graded free resolution of $\omega_R$ have entries in $\mathfrak{m}$, we see that $\omega_R$ has a $(-1)$-linear resolution. In particular,

$$\operatorname{reg}_R^T(\omega_R) = -1.$$

Thus it follows

$$\operatorname{reg}_R^T(\omega_R) = \operatorname{reg}_R^L(\omega_R) - \operatorname{reg}_R^L(R).$$

Assume that $\operatorname{pd}_R(\omega_R) < \infty$. Then it follows from the Auslander–Buchsbaum formula that $\operatorname{pd}_R(\omega_R) = \operatorname{depth}(R) - \operatorname{depth}(\omega_R) = 0 - 0 = 0$. Hence $\omega_R$ would be free which is not possible. We see that $\operatorname{pd}_R(\omega_R) = \infty$. $\square$

So it is still interesting to understand better the modules for which the extremal cases of Theorem 3.1 hold and we end this section with the following questions.

**Question 4.4.** *Let $R$ be a positively graded $K$-algebra. Can one characterize those finitely generated graded $R$-modules $M$ such that* $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$?

The other inequality $\operatorname{reg}_R^L(M) = \operatorname{reg}_R^T(M) + \operatorname{reg}_R^T(K)$ is only interesting for $R$ a Koszul algebra. Thus one might ask:

**Question 4.5.** *Let $R$ be a Koszul-algebra. Can one characterize those finitely generated graded $R$-modules $M$ such that* $\operatorname{reg}_R^L(M) = \operatorname{reg}_R^T(M)$?

## 5. Concluding examples

Let $R$ be a standard graded $K$-algebra and $M$ be a finitely generated graded $R$-module. In Theorem 3.1 we proved that

$$\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) \leqslant \operatorname{reg}_R^T(M) \leqslant \operatorname{reg}_R^L(M) + \operatorname{reg}_R^T(K).$$

We saw that if $\operatorname{pd}_R(M) < \infty$, then $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R) = \operatorname{reg}_R^T(M)$ is satisfied. For $R$ Koszul and $M = K$ we see that $\operatorname{reg}_R^T(M) = \operatorname{reg}_R^L(M)$ is true. Now it is a natural question whether in principle all values between $\operatorname{reg}_R^L(M) - \operatorname{reg}_R^L(R)$ and $\operatorname{reg}_R^L(M)$ are possible for the number $\operatorname{reg}_R^T(M)$. We will show that this is true over Koszul algebras. For this we need at first the following lemma.

**Lemma 5.1.** *Let $R$ be a Koszul-algebra and $M$ be a finitely generated graded $R$-module. If $M$ has a $(j-1)$-linear resolution, then $\mathfrak{m}M$ has a $j$-linear resolution. In particular, we have $\operatorname{reg}_R^T(\mathfrak{m}^j) = j$ for $j \geqslant 0$.*

**Proof.** We consider the short exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0$$

and the induced long exact Tor-sequence

$$\begin{aligned} \cdots &\to \operatorname{Tor}_{i+1}^R(M/\mathfrak{m}M, K) \to \operatorname{Tor}_i^R(\mathfrak{m}M, K) \to \operatorname{Tor}_i^R(M, K) \to \cdots \\ &\to \operatorname{Tor}_1^R(M/\mathfrak{m}M, K) \to \operatorname{Tor}_0^R(\mathfrak{m}M, K) \to \operatorname{Tor}_0^R(M, K) \to \operatorname{Tor}_0^R(M/\mathfrak{m}M, K) \to 0. \end{aligned}$$

Since $M$ has a $(j-1)$-linear resolution, we have in particular, that $M$ is generated in degree $j-1$. The module $M/\mathfrak{m}M$ is a finitely generated graded $K$-vector space. Hence

$$M/\mathfrak{m}M \cong \bigoplus K(-j+1)$$

and this is also an isomorphism of graded $R$-modules. The minimal graded free resolution of $M/\mathfrak{m}M$ is a direct sum of the linear minimal graded free resolutions of $K$ shifted by $j-1$. Thus we see that $M/\mathfrak{m}M$ has an $(j-1)$-linear resolution. For $k \neq j-1$ we obtain

$$\operatorname{Tor}_i^R(M/\mathfrak{m}M, K)_{i+k} = 0.$$

Considering again the long exact Tor-sequence above in degree $i+k$ for $k > j$ we get

$$\cdots \to \operatorname{Tor}_{i+1}^R(M/\mathfrak{m}M, K)_{i+1+k-1} \to \operatorname{Tor}_i^R(\mathfrak{m}M, K)_{i+k} \to \operatorname{Tor}_i^R(M, K)_{i+k} \to \cdots$$

and therefore

$$\operatorname{Tor}_i^R(\mathfrak{m}M, K)_{i+k} = 0.$$

We have $\operatorname{Tor}_i^R(\mathfrak{m}M, K)_{i+k} = 0$ for $k < j$ because $\mathfrak{m}M$ is generated in degrees $\geqslant j$. Thus we get that $\mathfrak{m}M$ has a $j$-linear resolution over $R$.

Since $R$ is Koszul, $K = R/\mathfrak{m}$ has a 0-linear resolution over $R$ which is equivalent to the fact that $\mathfrak{m}$ has a 1-linear resolution over $R$. An induction on $j \geqslant 1$ yields that $\mathfrak{m}^j$ has a $j$-linear resolution over $R$. $\square$

**Example 5.2.** Let $R$ be a Koszul algebra such that $\operatorname{depth}(R) > 0$ and $r = \operatorname{reg}_R^L(R) > 0$. Then we have for $0 < j < r$ that

$$0 = \operatorname{reg}_R^L(\mathfrak{m}^j) - r < \operatorname{reg}_R^T(\mathfrak{m}^j) = j < r = \operatorname{reg}_R^L(\mathfrak{m}^j).$$

For example consider the $d$th Veronese subring $S^{(d)}$ of a standard graded polynomial ring $S = K[x_1, \dots, x_n]$ for some integer $d > 0$ (i.e. $S^{(d)}$ is the graded $K$-algebra with $(S^{(d)})_i = S_{id}$ for $i \geqslant 0$). For $d \gg 0$ we have that $S^{(d)}$ is Koszul, $\operatorname{depth}(S^{(d)}) > 0$ and $\operatorname{reg}_{S^{(d)}}^L(S^{(d)}) > 0$.

**Proof.** It follows from Lemma 5.1 that

$$\operatorname{reg}_R^T(\mathfrak{m}^j) = j.$$

To determine $\operatorname{reg}_R^L(\mathfrak{m}^j)$ we consider the short exact sequence

$$0 \to \mathfrak{m}^j \to R \to R/\mathfrak{m}^j \to 0.$$

The induced long exact local cohomology sequence is

$$\begin{aligned} 0 &\to H^0_{\mathfrak{m}}(\mathfrak{m}^j) \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(R/\mathfrak{m}^j) \to \cdots \\ &\to H^i_{\mathfrak{m}}(\mathfrak{m}^j) \to H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R/\mathfrak{m}^j) \to \cdots. \end{aligned}$$

Observe that $R/\mathfrak{m}^j$ has finite length, is therefore zero-dimensional and we have that $H^i_{\mathfrak{m}}(R/\mathfrak{m}^j) = 0$ for $i > 0$. Since depth $R > 0$ we have $H^0_{\mathfrak{m}}(R) = 0$. Hence

$$H^0_{\mathfrak{m}}(\mathfrak{m}^j) \subseteq H^0_{\mathfrak{m}}(R) = 0.$$

Considering again the long exact local cohomology sequence we have

$$H^i_{\mathfrak{m}}(\mathfrak{m}^j) \cong H^i_{\mathfrak{m}}(R)$$

for $i = 0$ and $i > 1$. Moreover, the following sequence is exact:

$$0 \to H^0_{\mathfrak{m}}(R/\mathfrak{m}^j) \to H^1_{\mathfrak{m}}(\mathfrak{m}^j) \to H^1_{\mathfrak{m}}(R) \to 0.$$

Let $k \geqslant j - 1$. Then we have

$$H^0_{\mathfrak{m}}(R/\mathfrak{m}^j)_k = (R/\mathfrak{m}^j)_k \begin{cases} = 0 & \text{for } k > j-1, \\ \neq 0 & \text{for } k = j-1. \end{cases}$$

Thus

$$H^1_{\mathfrak{m}}(R)_k = 0 \quad \text{for } k > r$$

and we see that

$$\operatorname{reg}_R^L(\mathfrak{m}^j) = \max\{j, r\}.$$

For $0 < j < r$ we obtain the desired equalities

$$0 = \operatorname{reg}_R^L(\mathfrak{m}^j) - r < \operatorname{reg}_R^T(\mathfrak{m}^j) = j < r = \operatorname{reg}_R^L(\mathfrak{m}^j).$$

Now let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring. It is well known that for $d \gg 0$ the $d$th Veronese subring $S^{(d)}$ of $S$ is Koszul. (E.g. see [3] or [8].) The number $\operatorname{reg}^L_{S^{(d)}}(S^{(d)})$ coincides with $\operatorname{reg}^L_T(S^{(d)}) = \operatorname{reg}^T_T(S^{(d)})$ where $T$ is some polynomial ring such that $S^{(d)} = T/J$ for some graded ideal $J$ containing no linear forms. But $J$ is generated in degree 2 since $S^{(d)}$

is Koszul. Hence $\operatorname{reg}_T^T(S^{(d)}) \geqslant 1 > 0$. Since $S^{(d)}$ is Cohen–Macaulay of dimension $n$ (e.g. see [4, Exercise 3.6.21]) we have in particular $\operatorname{depth}(S^{(d)}) > 0$. This shows that we can apply the example to the $K$-algebra $S^{(d)}$. $\square$

## References

[1] L.L. Avramov, D. Eisenbud, Regularity of modules over a Koszul algebra, J. Algebra 153 (1) (1992) 85–90.
[2] L.L. Avramov, I. Peeva, Finite regularity and Koszul algebras, Amer. J. Math. 123 (2) (2001) 275–281.
[3] J. Backelin, R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures Appl. 30 (1985) 85–97.
[4] W. Bruns, J. Herzog, Cohen–Macaulay Rings, rev. ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, 1998.
[5] M. Chardin, On the behavior of Castelnuovo–Mumford regularity with respect to some functors, preprint, arXiv: math.AC/0706.2731, 2007.
[6] D. Eisenbud, Commutative Algebra. With a View Toward Algebraic Geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, 1995.
[7] D. Eisenbud, S. Goto, Linear free resolutions and minimal multiplicity, J. Algebra 88 (1984) 89–133.
[8] D. Eisenbud, A. Reeves, B. Totaro, Initial ideals, Veronese subrings, and rates of algebras, Adv. Math. 109 (2) (1994) 168–187.
[9] R. Fröberg, Determination of a class of Poincaré series, Math. Scand. 37 (1975) 29–39.
[10] J. Herzog, G. Restuccia, Regularity functions for homogeneous algebras, Arch. Math. 76 (2) (2001) 100–108.
[11] P. Jørgensen, Linear free resolutions over non-commutative algebras, Compos. Math. 140 (4) (2004) 1053–1058.
[12] S.B. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. 152 (1970) 39–60.
[13] J.E. Roos, Commutative non-Koszul algebras having a linear resolution of arbitrarily high order. Applications to torsion in loop space homology, C. R. Acad. Sci. Paris Sér. I 316 (11) (1993) 1123–1128.
[14] C.A. Weibel, An Introduction to Homological Algebra, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, 1995.